

Josephson effect in topological superconducting rings coupled to a microwave cavity

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We theoretically study a one dimensional p -wave superconducting mesoscopic ring interrupted by a weak link and coupled inductively to a microwave cavity. We establish an input-output description for the cavity field in the presence of the ring, and identify the electronic contributions to the cavity response and their dependence on various parameters, such as the magnetic flux, chemical potential, and cavity frequency. We show that the cavity response is 4π periodic as a function of the magnetic flux in the topological region, stemming from the so called fractional Josephson current carried by the Majorana fermions, while it is 2π periodic in the non-topological phase, consistent with the normal Josephson effect. We find a strong dependence of the signal on the cavity frequency, as well as on the parity of the ground state. Our model takes into account fully the interplay between the low-energy Majorana modes and the gaped bulk states, which we show is crucial for visualizing the evolution of the Josephson effect during the transition from the topological to the trivial phase.

I. INTRODUCTION

Majorana fermions are zero energy quasiparticles that are their own antiparticles. In condensed matter physics, they emerge as zero energy excitations in the so called topological superconductors, such as genuine p -wave superconductors¹⁻⁴, topological insulators in the presence of conventional superconductors⁵⁻⁷, nanowire in proximity to s -wave superconductor and subject to a magnetic field⁸⁻¹¹, in chains of magnetic atoms¹²⁻¹⁸. Majorana fermions in condensed matter physics were first predicted by Kitaev in Ref. 3 (see Ref. 19 for a review). They obey non-abelian statistics² in two dimensions, and they are robust against local perturbations that do not close the superconducting gap, which make them promising candidates as qubits for a functional topological quantum computer.

There are several physical manifestations associated with the presence of Majorana fermions, such as the zero bias anomaly in the topological region, the dependence of the Majorana energy splitting on the chemical potential, Zeeman splitting, or the size of the system, as well as the fractional Josephson effect and the non-abelian statistics pertaining to braiding of the Majoranas¹⁹. The fractional Josephson effect is one of the hallmarks of the Majorana fermions^{3,4,6,9,20}. It corresponds to 4π periodicity of the supercurrent in the superconductor phase difference across the weak link, as opposed to 2π periodicity for a usual Josephson effect in conventional s -wave superconductors, and it arises because of the degeneracy of the zero energy Majorana modes. In other words, because of such degeneracy, there is a coherent charge e transfer across the weak link, instead of $2e$ as in the normal Josephson effect. There are various ways to reveal the fractional Josephson effect, such as the measurement of the Shapiro steps in the $I - V$ characteristics of a voltage biased junction^{21,22}, by employing a resistive shunted SQUID setup²³, or by performing microwave spectroscopy of the junction. The latter method is particularly interesting as it allows to access also the

exited states of the junction. There are several theoretical works that study the interplay of the Majorana fermions physics and microwaves in a superconducting cavity QED setup²⁴⁻³⁴. In contrast to the usual electronic methods, this approach is unique in that it can be totally non-invasive, i.e. it does not alter the electronic system, a crucial feature in view of using these excitations for quantum computing. However, almost all of these studies deal with effective models that consider *only* the Majorana fermions interacting with the cavity field, which cannot account for the evolution of the Majorana fermions through the topological transition from a non-trivial to a trivial topological superconductor. More specifically, the Majorana fermions emerge as edge modes in a topological superconductor: below the phase transition the system is non-trivial, and shows edge modes, while above the phase transition the system is trivial, and is has no edge modes. Majorana fermions are thus intimately related to the bulk physics, and are not simply impurity states in the superconductor. The fractional Josephson effect disappears above the phase transition, where it becomes the normal 2π periodic one. In order to account for the evolution of the system from topological to trivial, associated with the fractional and normal Josephson effect, respectively, one needs to consider the Majorana and the bulk states on the same footing. In this paper, we do precisely that in the context of the cavity QED setup that was first proposed in Ref. 35 and experimentally implemented in Ref. 36. The recent progress in engineering sensitive superconducting cavities makes it possible to implement on-chip mesoscopic circuits coupled to high finesse resonators³⁷.

In this paper, we study a mesoscopic superconducting ring with a weak link threaded by a magnetic flux and coupled to the cavity. We consider an inductive coupling between the mesoscopic ring and the cavity³⁸. We show that the electronic susceptibility has a different periodicity as a function of the magnetic flux in the topological and non-topological regions, thus allowing us to probe whether the wire is in topological or non-

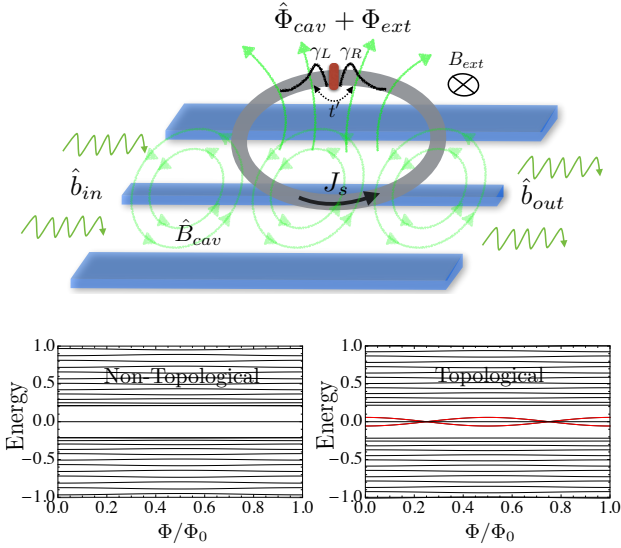


Figure 1: Top: A sketch of the combined topological superconducting ring inserted in a microwave cavity. The magnetic field of the cavity \hat{B}_{cav} pierces the ring and gives rise to a fluctuating magnetic flux. A perpendicular, external magnetic field B_{ext} is also applied, so that the total flux felt by the electrons in the ring is $\Phi_{tot} = \Phi_{ext} + \hat{\Phi}_{cav}$. The ring is interrupted by a weak link, that pertains to the hopping strength $t' \ll t$ (t is the hopping strength within the ring, see the text). In the topological region, there are Majorana fermions on the left (γ_L) and on the right (γ_R), and a persistent current J_s is flowing within the ring. The cavity is probed in the input-output method, and the output signal \hat{b}_{out} is to be compared with the input one \hat{b}_{in} . Bottom: Spectrum of the ring as a function of the flux Φ_{ext} in the non-topological (left) and topological (right) regions, respectively. In red are depicted the Majorana energy levels that show oscillations with a period $\Phi_0/2$.

topological region: 2π ($\Phi_0/2 = h/2e$) periodicity of the Josephson current²⁰ corresponds to the non-topological region, while 4π ($\Phi_0 = h/e$) periodicity of the observables corresponds to the topological region, when Majorana fermions emerge in the system.

The paper is organized as follows: in Sec. II we introduce the p -wave superconducting model and the Hamiltonians. Specifically, we consider the Kitaev model in the topology of a ring with a weak link. Moreover, we assume the ring to be threaded by both dc and ac magnetic fluxes. In Sec. III we analyze the coupling between the cavity and the superconducting ring, as well as calculate the response of the cavity due to this coupling. In Sec. IV we calculate the reactive and dissipative response of the cavity, and show the numerical results. Finally, in Sec. V we end up with conclusions and provide some outlook.

II. THE SYSTEM AND MODEL HAMILTONIAN

In Fig. 1, we present a sketch of the system under consideration: a topological superconducting ring which contains a weak link at some position coupled to a superconducting microwave cavity. The ring is subject to both a dc magnetic flux, and an ac (quantum) flux from the cavity, which gives rise to a coupling between the two systems, the ring (electrons) and cavity (photons). In this paper, we consider the Kitaev model Hamiltonian as describing the electronic p -wave superconductor. Such a model could be implemented experimentally utilizing a semiconducting nanowire with spin-orbit coupling in the proximity of s -wave superconductor and subject to a magnetic field^{8,9}. We assume the ring is interrupted at some point by a weak link that emulates the Josephson effect in a SIS junction. The external magnetic flux allows for a continuous phase change across the weak link, which controls the Josephson current. On the other hand, the fluctuating cavity flux couples to the flux-biased system and monitors its dynamics.

The general form of the combined Hamiltonian reads:

$$H_{sys} = H_{el} + H_{ph} + H_T, \quad (1)$$

$$H_{el} = - \sum_{j=1}^{N-1} (t e^{i\Phi_{jj+1}} c_j^\dagger c_{j+1} + \Delta e^{i\phi_{jj+1}} c_j c_{j+1} + \text{h.c.}) - \mu \sum_{j=1}^N c_j^\dagger c_j, \quad (2)$$

$$H_T = -t' e^{i\Phi_{N1}} c_N^\dagger c_1 + \text{h.c.}, \quad (3)$$

where H_{el} is the Hamiltonian of the p -wave superconductor, H_T is the tunneling Hamiltonian between the ends in the presence of the magnetic flux, and $H_{ph} = \omega_c a^\dagger a$ is the Hamiltonian of a single mode cavity, where ω_c is the cavity frequency and a^\dagger (a) is the photon creation (annihilation) operators, respectively. Also, t is the hopping parameter, Δ is the p -wave superconducting pairing potential, μ is the chemical potential, and N is the total number of sites. c_j^\dagger (c_j) is the electron creation (annihilation) operators, respectively, at the site j . Moreover, Φ_{jj+1} and ϕ_{jj+1} are the phases on the tunneling matrix elements t and the superconducting pairing Δ at positions j caused by the external fluxes. They read:

$$\Phi_{jj+1} = \frac{e}{\hbar} \int_j^{j+1} dx A(x) \equiv \frac{2\pi\Phi_{tot}}{N\Phi_0}, \quad (4)$$

$$\phi_{jj+1} = \frac{4\pi\Phi_{tot}j}{N\Phi_0}. \quad (5)$$

The form of the second term (the phase on the pairing Δ) can be found from utilizing the following assumption: the p -wave pairing in the Kitaev chain is assumed to be induced by an s -wave superconductor underneath the wire that is interrupted at the link. In such a case, we can consider that the instantaneous supercurrent flowing

through the superconductor vanishes, namely:

$$J_s = \frac{2e}{m} |\psi|^2 (\hbar \nabla \phi - 2eA) \equiv 0, \quad (6)$$

where m , $|\psi|^2$, and ϕ are the electronic mass, the density of superconducting electrons in the s -wave superconductor, and its phase, respectively. The latter is directly imprinted into the wire by means of proximity effect. We can easily solve this differential equation, which then gives the phase on the p -wave superconductor²⁰ $\phi(j) \equiv \phi_{jj+1} = 4\pi \Phi_{tot} j / N \Phi_0$.

We note that the system is in topological (non-topological) phase when $|\mu| < 2t$ ($|\mu| > 2t$), and it supports Majorana zero energy end modes (no Majorana end modes).

The magnetic flux contains both the dc flux Φ_{dc} , as well as a time-dependent component $\hat{\Phi}_{ac}(t)$, or $\Phi = \Phi_{dc} + \hat{\Phi}_{ac}(t)$. We assume that the ac flux is due to the cavity photons, and it is given by $\hat{\Phi}_{ac} = i\lambda(a - a^\dagger)$, with λ being the coupling constant between the ring and the cavity. For the moment, we keep this parameter arbitrary, and discuss its possible values later on.

We can rewrite the full Hamiltonian in a simpler form by performing a unitary transformation $U = \exp[-2\pi i \sum_{j=1}^N (j - 1/2) n_j \Phi_{dc} / N \Phi_0]$ so that the entire dependence on the dc flux Φ_{dc} is transferred to the weak link tunneling Hamiltonian H_T , or:

$$H_T \rightarrow U^\dagger H_T U = -t' e^{i2\pi \Phi_{dc} / \Phi_0} e^{i2\pi \hat{\Phi}_{cav} / N \Phi_0} c_N^\dagger c_1 + \text{h.c.} \quad (7)$$

We mention that such a transformation can only gauge away the dc flux (i.e. transfer it entirely to t'), while the ac (or cavity component) cannot be gauged away in a simple way because such a transformation would not commute with the photonic Hamiltonian H_{ph} . However, we assume that $\hat{\Phi}_{ac} \ll \Phi_0$, so that we can expand the exponentials in $\hat{\Phi}_{ac}$ up to second order in this quantity. With these approximations, the final Hamiltonian can be written as the sum of the electrons (in the presence of the dc flux), the photons, and their mutual interaction:

$$H_{sys} \approx H_{el} + H_T + H_{ph} - \hat{\Phi}_{ac} I + \frac{1}{2} (\hat{\Phi}_{ac})^2 D, \quad (8)$$

$$I = \frac{2\pi i}{N \Phi_0} \sum_{j=1}^{N-1} \left[t c_j^\dagger c_{j+1} + 2j \Delta c_j c_{j+1} + t' e^{i2\pi \Phi_{dc} / \Phi_0} c_N^\dagger c_1 - \text{h.c.} \right], \quad (9)$$

$$D = - \left(\frac{2\pi i}{N \Phi_0} \right)^2 \sum_{j=1}^{N-1} \left[t c_j^\dagger c_{j+1} + (2j)^2 \Delta c_j c_{j+1} + t' e^{i2\pi \Phi_{dc} / \Phi_0} c_N^\dagger c_1 + \text{h.c.} \right]. \quad (10)$$

This is the Hamiltonian that we will be utilizing in the following to describe the response of the cavity, up to second order in the electron-cavity coupling.

Such configuration (ring geometry) have been studied in the past in the context of persistent currents in *normal*

rings, and it was shown to be physically equivalent to a SNS junction, where the low-energy system is described by the so called Andreev levels. In both cases, the spectrum is quantized due to the finite size of the ring (N system), or the finite size of the N part (SNS junction). We mention that for the SNS junction, if the length of the normal part $L \ll \xi$ (ξ is the coherence length), the spectrum shows a gap that is modulated by the external flux, and can even vanish. This gap, or mini gap, is usually much smaller than the bulk superconducting gap, and these levels are responsible for all the low-energy transport properties of the junction. The Andreev levels in the presence of the flux give rise to persistent currents which, for the state n with energy ϵ_n reads:

$$i_n = - \frac{\partial \epsilon_n}{\partial \Phi_{dc}}. \quad (11)$$

The Josephson current carried by these states is then given by $I_J = \sum_n f_n i_n$, where f_n are the occupations (Fermi-Dirac) of the level n .

III. ELECTRONIC SUSCEPTIBILITY AND CAVITY RESPONSE

A. Theoretical approach to susceptibility

In this section, we discuss the cavity response in the presence of the p -wave superconductor, and the dependence on various electronic parameters, such as the chemical potential and the applied magnetic flux Φ_{dc} .

There are two ways to approach the problem: utilizing a quantum description, namely treating the photons as quantum objects, or assuming the ac component of the flux is a classical oscillating quantity, of the form $\Phi_{ac} = \Phi_{ac}(0) \sin(\omega t)$. In this paper, we will employ the first method, but compare with the expected results utilizing a classical description. The reason for doing so is that the first method is suited to also enter the quantum regime, where the coupling between the photons and the electronic system is strong and one needs to employ a polaritonic-like description. While such a case is left for a future study, we believe that the formalism developed in this paper will be of great usefulness.

In previous works^{34,39}, we derived explicitly the response of the cavity due to its coupling to an electronic system. Here we give a brief summary of the derivation, and depict the results for the particular type of couplings found here. For more details on the general derivation we refer the reader to Ref. 34. The basic idea is the following: the cavity is probed in transmission, namely photons are sent toward it, and the outcome photons are collected. By comparing the input and output signals, one can infer the properties of the electronic system. One can start from the equation of motion for the cavity field⁴⁰:

$$\dot{a} = \frac{i}{\hbar} [H_{sys}, a] - \frac{\kappa}{2} a - \sqrt{\kappa} b_{in}, \quad (12)$$

for the input field, and

$$\dot{a} = \frac{i}{\hbar} [H_{sys}, a] + \frac{\kappa}{2} a - \sqrt{\kappa} b_{out}, \quad (13)$$

for the output field, with $b_{out} = b_{in} + \sqrt{\kappa} a$. Here, κ , b_{in} , and b_{out} are the escape rate of the photons from the cavity, the input, and the output photonic fields, respectively, while H_{sys} is the total Hamiltonian of the cavity and electrons, including their interaction. This is in general a very complicated problem to solve, as the photonic and electronic dynamics are correlated and thus there are back-action effects between the two systems. However, in the weak coupling limit the problem can be solved iteratively. Note that the second order contribution to the electron-photon coupling can be written as $(a^\dagger)^2 + a^2 + 2n_{ph} + 1$, with $n_{ph} \equiv a^\dagger a$, and we see that the third and fourth terms renormalize the cavity frequency and the weak tunneling amplitude, respectively. The first two terms correspond to creation or annihilation of two photons, and we can disregard them in the weak coupling limit. With this approximation, we can insert this expression into the equation of motion for the photonic operator in Eqs. (12) and (13), switch to the frequency domain, and solve these equations iteratively up to second order in λ . By taking the average over the unperturbed (or uncoupled to the cavity) electronic Hamiltonian, we obtain:

$$\begin{aligned} -i\omega a &= -i(\omega_c + \lambda^2 \langle D \rangle) a - \frac{\kappa}{2} a - \sqrt{\kappa} b_{in} \\ &- i\lambda^2 \Pi(\omega_c) a + \lambda \langle I \rangle, \end{aligned} \quad (14)$$

where $\Pi(\omega) = \int dt \exp(i\omega t) \Pi(t)$ with

$$\Pi(t) = -i\theta(t) \langle [I(t), I(0)] \rangle. \quad (15)$$

Note that all the expectation values are taken over the electronic system in the absence of the cavity. Let us discuss briefly the resulting equation of motion and further approximations. We assume the input field b_{in} contains a large number of photons, and thus we can neglect the last term that acts as an extra input source (and which is entirely due to the current flowing through the weak link). The opposite limit, when the input field is only due to vacuum fluctuations, can in principle lead to emission of photons due to current fluctuations in the ring. Such physics have been studied in connection with light emission by voltage biased tunnel and Josephson junctions, and is beyond the scope of our paper.

With this, we see that the coupling to the superconductor affects the cavity in two ways: by changing its frequency, and by changing its quality factor (or the escape rate κ). More precisely, we have:

$$\delta\omega = \lambda^2 [\langle D \rangle + \mathcal{R}e\Pi(\omega_c)], \quad (16)$$

$$\kappa' = \kappa - \lambda^2 \mathcal{I}m\Pi(\omega_c). \quad (17)$$

In Ref. 34,39 we showed explicitly that the transmission τ of the cavity can be written as follows:

$$\tau \equiv \frac{b_{out}}{b_{in}} = \frac{\kappa}{-i(\omega - \omega_c) + \kappa + i\lambda^2 \Pi_{tot}(\omega_c)}, \quad (18)$$

where $\Pi_{tot}(\omega_c) = \Pi(\omega_c) + \langle D \rangle$. It is instructive to rewrite the two terms in more explicit forms:

$$\langle D \rangle = \sum_n f_n \langle n | D | n \rangle, \quad (19)$$

$$\Pi(\omega) = \sum_{n \neq m} |\langle n | I | m \rangle|^2 \frac{f_n - f_m}{-(\epsilon_n - \epsilon_m) + \omega + i\eta}, \quad (20)$$

where f_n is the occupation of the electronic energy state ϵ_n , and η is a small number that account for the dissipation. We can rewrite $\Pi_{tot}(\omega)$ in a more transparent form by utilizing the following sum rule^{41,42}:

$$\langle n | D | n \rangle + \sum_{m \neq n} \frac{|\langle n | I | m \rangle|^2}{\epsilon_n - \epsilon_m} = \frac{1}{2} \frac{\partial^2 \epsilon_n}{\partial \Phi^2}, \quad (21)$$

so that we obtain

$$\begin{aligned} \Pi_{tot}(\omega) &= \frac{\partial J_s}{\partial \Phi} + \sum_n \frac{\partial f_n}{\partial \Phi} \frac{\partial \epsilon_n}{\partial \Phi} \\ &- \omega \sum_{n \neq m} \frac{f_n - f_m}{\epsilon_n - \epsilon_m} \frac{|\langle n | I | m \rangle|^2}{(\epsilon_n - \epsilon_m) - \omega - i\eta}, \end{aligned} \quad (22)$$

where $J_s = \sum_n f_n i_n$ is the persistent (or Josephson) current flowing through the ring in the presence of a *dc* flux. In this form, the expression for the susceptibility that is measured by the cavity (via its transmission) has the same form as the one derived previously for normal⁴¹ and for superconducting rings^{42,43}. However, in those previous works they utilized a classical description of the *ac* perturbation. Moreover, they assumed the electronic system is not closed, but coupled to an external bath which leads to a finite width of the electronic levels. To make correspondence between their results and ours in such a case, we can write:

$$\begin{aligned} \Pi_{tot}(\omega) &= \frac{\partial J_s}{\partial \Phi} + \frac{\omega}{\omega + i\gamma} \sum_n \frac{\partial f_n}{\partial \Phi} \frac{\partial \epsilon_n}{\partial \Phi} \\ &- \omega \sum_{n \neq m} \frac{f_n - f_m}{\epsilon_n - \epsilon_m} \frac{|\langle n | I | m \rangle|^2}{(\epsilon_n - \epsilon_m) - \omega - i\gamma}, \end{aligned} \quad (23)$$

where γ is the relaxation rate of the levels (assumed, for simplicity, to be the same for all levels). We see that, in principle, there are three contributions to the cavity response: the persistent current or non-dissipative, the diagonal or the decay of the levels, and the non-diagonal stemming from the usual Kubo contribution, respectively. In the following, we assume the zero temperature limit ($T = 0$), which in turn implies that the second term vanishes. The first term is independent on the frequency ω , and thus any dependence on this parameter will be due to the last term.

For the classical derivation of the response we refer the reader to Refs. [41,42,44]. A few comments are in order. In the case of a SNS junction, the entire contribution to the cavity signal is coming from the sub gap

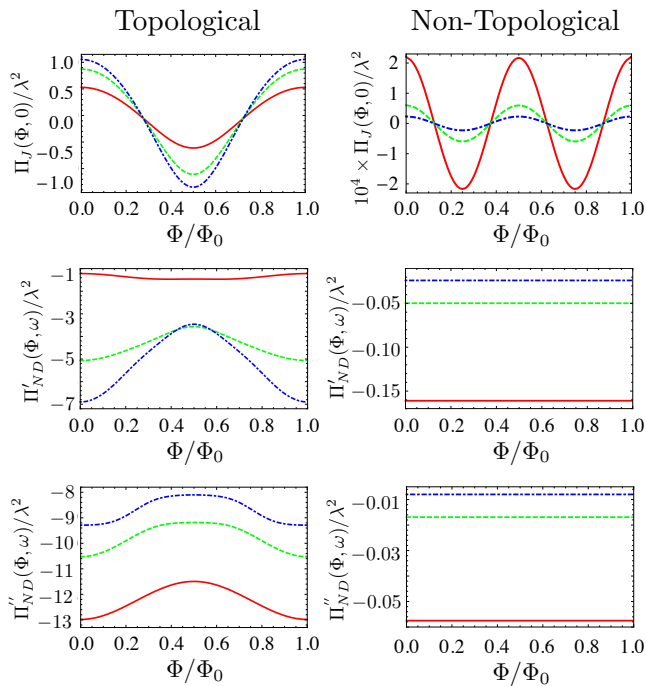


Figure 2: Left row (right row), from top to bottom: Dependence of the Josephson, non-diagonal real, non-diagonal imaginary susceptibilities respectively, in the topological (non-topological) region on the magnetic flux Φ/Φ_0 for $f_M = 0$. The red, green, and blue lines correspond to $\mu = -1.5$ ($\mu = -2.5$), $\mu = -1$ ($\mu = -3$), and $\mu = -0.5$ ($\mu = -3.5$), respectively. The phase transition point is at $\mu = -2$. The parameters for the plots are $\Delta = 0.1$, $t' = 0.2$, $N = 50$, and all energies are expressed in terms of $t = 1$.

Andreev states in the normal region, while any bulk effects can be neglected since the bulk gap is much larger than the minigap. That need not be the case for a p -wave superconductor, as this system can suffer a topological transition and close its bulk gap, which in turn should affect strongly the cavity response as a function of the magnetic flux. Thus, the energy levels in the expression for the susceptibility are the ones that correspond to the entire bulk spectrum, and not only the mini gap states.

B. Flux dependence of the cavity response

In Fig. 2, we plot separately the different contributions to the susceptibility measured by the cavity, as a function of the external flux: the Josephson current contribution (top), the real (middle) and the imaginary (bottom) non-diagonal contributions both in the topological (left column) and the non-topological phases (right column). We see that in the topological regime, all susceptibilities are Φ_0 periodic, with large amplitudes variations. The amplitude of the Φ_0 oscillations, however, diminishes as the system approaches the topological transition, and disappear after this transition, with only the $\Phi_0/2$ becoming

manifest. Such $\Phi_0/2$ oscillations pertain to the normal Josephson effect, and are much smaller in amplitude, as can be seen from the top plot on the right column. We note that such oscillations exist also for the non-diagonal terms, but they are too small to be depicted (of the order 10^{-6}).

Let us now discuss the various terms contributing to the total susceptibility, as well as the effect of Majorana fermions on these quantities. As mentioned already, in the topological regime, there are Majorana fermions emerging at the edge of the ring, and which are coupled via the weak link. In a low-energy description, these Majoranas are responsible for the so called fractional Josephson effect, one of the hallmarks of the Majorana fermions physics. However, the response of the cavity is sensitive not only to the presence of Majoranas, but also to the interplay between these excitations and the bulk states of the superconductor, in particular close to the topological phase transition. The first term is insensitive to such transitions, as it is given by the derivative of the persistent current with respect to the applied flux. The second term instead contains the matrix elements between the Majorana and the bulk states, it has both a real and imaginary parts, and depends on the cavity frequency ω_c . We can thus decompose it as follows:

$$\Pi_{tot}(\omega) = \Pi_{BB}(\omega) + \Pi_{BM}(\omega) + \Pi_{MM}(\omega), \quad (24)$$

where Π_{BB} is the part of the susceptibility that comes from the transitions between (different) bulk levels, Π_{BM} corresponds to the transitions between bulk and Majorana levels while Π_{MM} corresponds to the transitions between Majorana levels and $\Pi_{MM}(\omega) \equiv 0$ in the present setup. Note that in the non-topological regime $\Pi_{BM} = 0$, as there are no Majorana states present.

To ascertain for the physical meaning to the oscillations of the susceptibility, it is instructive to cast this quantity in the following form:

$$\Pi_{tot}(\omega, \phi) = \sum_{j=0}^{\infty} a_j \cos(2\pi j \Phi/\Phi_0 + \delta\phi_j), \quad (25)$$

where a_j are (complex) Fourier coefficients of the expansion, and $\delta\phi_j$ are phase shifts, all these quantities being functions of the chemical potential μ and the gap Δ . In the topological phase, the dominant terms are a_0 and a_1 , namely those corresponding to constant and Φ_0 oscillations, respectively. In the topologically trivial regime instead, $a_1 = 0$, and thus the usual Josephson effect dominates. Moreover, we expect that all the coefficients with $j = 2p + 1$, $p = 1, 2, \dots$ to be zero above the topological transition (which does not host any Majoranas), while below such coefficients are in principle non-zero. We stress that the Josephson current has also a similar dependence on the applied flux: for a p -wave superconductor, the current $I_J(\Phi) \propto \sin(4\pi\Phi/\Phi_0)$ in the non-topological region (normal Josephson effect), while this is $I_J(\Phi) \propto \sin(2\pi\Phi/\Phi_0)$ in the topological region (fractional Josephson effect).

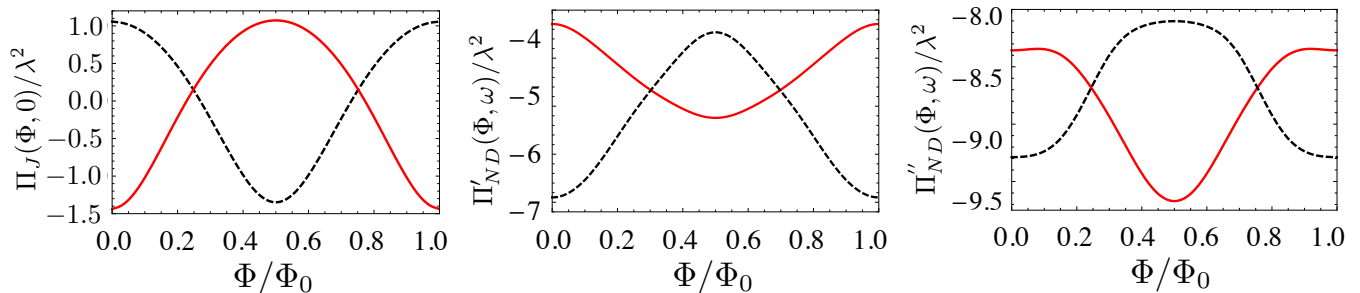


Figure 3: Dependence of the Josephson, non-diagonal real, and non-diagonal imaginary susceptibilities, respectively, on the magnetic flux Φ and parity f_M . The dashed-black (full-red) curves correspond to parity $f_M = 0$ ($f_M = 1$). The oscillations of the susceptibilities for the two parties are both shifted by Φ_0 , as well as asymmetric in amplitudes (once superimposed). The parameters for the plots are $\mu = -0.5$, $\Delta = 0.1$, $t' = 0.2$, $N = 50$, and all energies are expressed in terms of $t = 1$.

C. Parity dependence

In this section, we analyze the parity dependence of the electronic susceptibility, and implicitly of the cavity response. In Fig. 3, we plot the various contributions to the susceptibility in the topological regime for the two different parities of the ground state, $f_M = 0, 1$. By inspecting the plots, we can conclude that there are two main features that separate the two parities: first, the oscillations (or maxima) of the susceptibilities are shifted by Φ_0 , and second, there is a strong asymmetry between the curves for the two parities, especially in the non-diagonal contribution. The shift by Φ_0 can be explained by means of the fractional Josephson effect, which is due to the presence of Majorana bound states that give rise to coherent transfer of charge e or $-e$, depending on the parity, between the two bound states. That in turn gives rise to dissipationless currents opposite in sign for the two parities, and which coincide for $\Phi = (2n+1)\Phi_0/4$, as can be seen also from the plots. The asymmetry of the Josephson contribution can be explained by inspecting the expression for the supercurrent flow: $I_J = \sum_{n=bulk} i_n + (-1)^{f_M} i_M$, i.e. the bulk contribution is constant, as its parity is assumed unchanged, while the Majorana contribution depends on the parity of the ground state $f_M = 0, 1$. The non-diagonal susceptibility shows a much stronger asymmetry, and it is due to an interplay between the occupancy of the Majorana and the matrix elements of the current operator I between the Majorana and the bulk states, or simply by Π_{BM} given by:

$$\Pi_{BM}(\omega) = -\omega \sum_{n \neq M} \frac{f_n - f_M}{\epsilon_n - \epsilon_M} \frac{|\langle n | I | M \rangle|^2}{(\epsilon_n - \epsilon_M) - \omega - i\gamma}, \quad (26)$$

where M stands for the Majorana.

D. Frequency dependence

Finally, we discuss briefly the frequency dependence of the susceptibility (thus of the cavity response) for different parity and flux realizations, respectively. We note

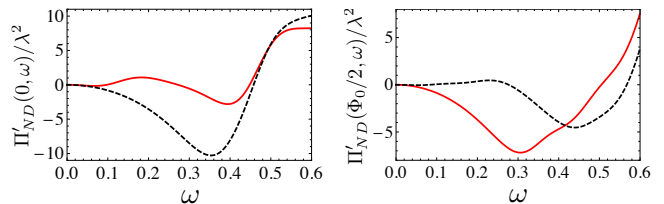


Figure 4: Dependence of the real part of the non-diagonal susceptibility on the frequency ω for two values of the external flux, $\Phi = 0$ (left) and $\Phi = \Phi_0/2$ (right), and for the two parity states $f_M = 0$ (dashed-black) and $f_M = 1$ (full-red) lines, respectively. The parameters for the plots are: $\mu = -0.5$, $\Delta = 0.1$, $t' = 0.5$, $N = 50$, and all energies are expressed in terms of $t = 1$.

that the Josephson contribution is frequency independent, and the only dependence on ω arises from the non-diagonal susceptibility. In Fig. 4, we plot the real part of the non-diagonal susceptibility as a function of the frequency in the topological regime, and for the two different parities. The non-monotonic behavior of the curves can be related to the resonance condition when the frequency ω matches the effective p -wave gap and the sign of the susceptibility changes. Interestingly, the dependence on ω is different for different parities $f_M = 0, 1$ and for different values of the magnetic flux Φ . Such features can again be traced back to the dependence of Π_{BM} on both the flux and the parity of the topological superconductor. We note, however, that the curves become identical (not shown) at flux values $\Phi = (2n+1)\Phi_0/4$, with $n = 0, 1, \dots$, at which instances the degeneracy of the Majorana energy levels is restored.

IV. CONCLUSIONS

In this paper, we studied theoretically a Kitaev ring interrupted by a weak link and pierced by a magnetic flux, and coupled inductively to a microwave cavity. We established an input-output description for the cavity transmission, and showed that the cavity response depends

strongly on various electronic parameters, such as the chemical potential, the magnetic flux, and the parity of the superconducting ground state. We found a 4π (2π) variation of the cavity response with respect to the external magnetic flux in the topological (trivial) phase, and related such dependence to the fractional (normal) Josephson effect. As opposed to previous works, our theory takes into account, on equal footing, the low-energy Majorana modes, the gaped bulk states, and the interplay between these states in the presence of the cavity. Such a description allows one to describe not only the superconducting ring deep in the topological regime, but also the topological transition, and the crossover from the fractional to the normal Josephson effect in the presence of a magnetic flux.

While our theory treats the Kitaev toy model, we stress that it has been shown theoretically that such a model can be emulated by the spectrum of a spin-orbit semiconducting ring subject to an external magnetic field and

coupled by the proximity effect to an s -wave superconductor. The main difference between such a model and the Kitaev chain is the number of states, which is doubled, as one needs to account for the spin degree of freedom. While such a study is beyond the scope of our paper, we strongly believe, based on our previous findings in Ref. 34, that our main results on the visualization of the transition from the fractional to the normal Josephson effect by utilizing a microwave cavity are robust.

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